Optimization

<u>Last time</u>: gradient = direction of maximal increase in f.

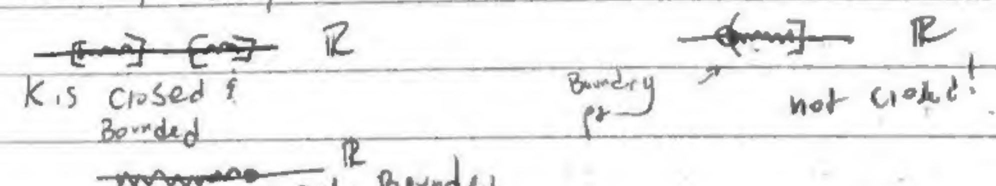
<u>A critical pt (CP)</u> of f is a point $\vec{p}$ in the domain of f where either ∇f @ $\vec{p} = 0$ or $\nabla f(\vec{p})$ does not exist

<u>prop (Fermat extremum thm)</u>: If f has a local extreme value @ p, then $\vec{p}$ is a crit. pt. of f

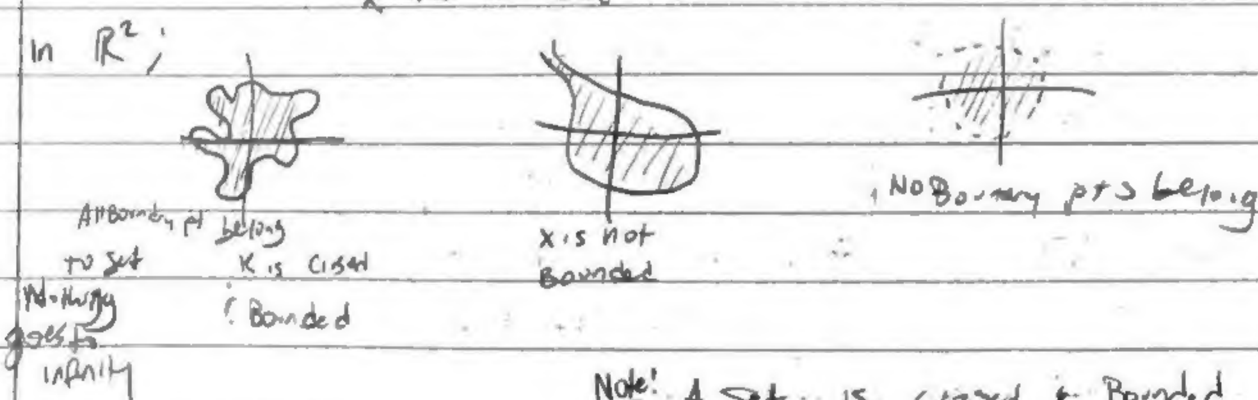
to do local optimization we also need:

<u>Prop (Extreme Value Theorem)</u>: if f is defined on a <u>closed</u> & <u>bounded</u> subset $K \subseteq \mathbb{R}^n$, then f attains its global extrema (on K)

<u>What is "closed & bounded"?</u>: In $\mathbb{R}^1$ a set is closed & bounded iff it is a union of finitely many closed and bounded intervals



In $\mathbb{R}^2$:



<u>Note</u>: A set is closed & Bounded when it is bounded and its "separating points from the rest of $\mathbb{R}^n$" are all in the set →

This Suggests A method for optimizing global values on a closed and bounded Subset

Alg (compact set method): Let f be a function defined on a closed & Bounded Subset K. To compute global extrema of f on K:

① compute critical points of f within K

② Compute the max and min of f on those (CP)s

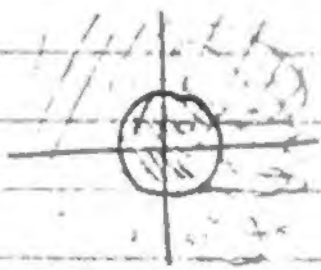
③ optimize along the boundary curve

The max/min values are global extreme values of f on K

Ex: find global extrema of $f(x,y) = xy^2$ on $K = \{(x,y): 0 \leq x ; 0 \leq y ; x^2+y^2 \leq 3\}$

SoL: First compute Critical points

Picture



$\nabla f = \langle y^2, 2xy \rangle$

$\nabla f = \vec{0}$ iff $\langle y^2, 2xy \rangle = \vec{0}$

iff $\begin{cases} y^2=0 \\ 2xy=0 \end{cases}$ iff $\begin{cases} y=0 \\ x=0 \,\&\, y=0 \end{cases}$

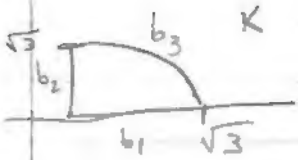
So iff $y=0$

So K =

$y=0$

Note: in this example, Boundary points contain the (CP)s that intersect K.... So Analyzing the Boundary will Also Analyze the (CP) (Skipping Step 2 & Just compute Step 3)

~~Now lets Analyze the Boundary~~



Parametrize the Boundry Curves like so:

$$b_1(t) = (t, 0) \text{ on } 0 \le t \le \sqrt{3}$$
$$b_2(t) = (0, t) \text{ on } 0 \le t \le \sqrt{3}$$
$$b_3(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t) \quad 0 \le t \le \frac{\pi}{2}$$

Next we optimize $f(b_i(t))$ for each Boundry piece

On $b_1(t)$: $f(b_1(t)) = f(t,0) = t \cdot 0^2 = 0$ — Both the Abs max and abs min in B_1 & B_2

On $B_2(t)$: $f(b_2(t)) = f(0,t) = 0 \cdot t^2 = 0$

On $B_3(t)$: $f(b_3(t)) = f(\sqrt{3}\cos t, \sqrt{3}\sin t)$

$$= \sqrt{3}\cos t \cdot \sqrt{3}\sin t^2$$
$$= 3\sqrt{3}\cos t \sin^2 t$$
$$= g(t)$$

So $g'(t) = 3\sqrt{3}\left((-\sin t)\sin^2 t + \cos t\,(2\sin t \cos t)\right) =$

$g't = 3\sqrt{3}\sin t\,(2\cos^2 t - \sin^2 t)$

$\therefore g'(t) = 0$ iff $\sin(t) = 0$ or $2\cos^2(t) - \sin^2(t) = 0$

$$2\cos^2 t = \sin^2 t$$

$\sin t$ & $\cos t$ cannot both be 0 for the same value of t

Divide by $\cos^2 t$

iff $\sin t = 0$ or $2 = \tan^2 t$

iff $\sin(t) = 0$ or $\tan(t) = \pm\sqrt{2}$

iff $t = K\pi$ or $t = \mathrm{atan}(\sqrt{2})$ or $t = \mathrm{atan}(-\sqrt{2})$

for some integer K

$\longrightarrow$

$0 \le t \le \frac{\pi}{2}$

iff $t = K\pi$ or $t = \operatorname{atan}(\sqrt{2})$ or $t = \operatorname{atan}(-\sqrt{2})$

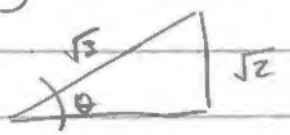
iff $t = 0$ or $t = \operatorname{atan}(\sqrt{2})$

$\cancel{\operatorname{atan}(-\sqrt{2})} \le 0$ So Reject

$\therefore$ testing $g(t)$ at the Boundary / crit. pt.:

$g(0) = 0$, $g(\frac{\pi}{2}) = 0$

$g(\arctan(\sqrt{2})) = 3\sqrt{3}\cos(\operatorname{atan}(\sqrt{2}))\sin^2(\operatorname{atan}(\sqrt{2})) > 0$



$\theta = \operatorname{atan}(\sqrt{2})$

$\tan\theta = \frac{\sqrt{2}}{1}$

$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

$= 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2$

$= 2 \longrightarrow 2 > 0$

$\therefore$ ABS max on K is 2 and the abs min on K is 0 for f $\boxed{\checkmark}$

Q?: How Do we make Analogs for the 1st & 2nd Derivative test in Calc 3?

First Derivative test: Let f be differentiable at $\vec{p}$

① if $D_{\vec{u}} f(\vec{p} + \varepsilon\vec{u}) > 0$ for All sufficient small $\varepsilon > 0$ and all unit vectors $\vec{u}$, then f has a local min @ $\vec{p}$

② If $D_{\vec{u}} f(\vec{p} + \varepsilon\vec{u}) < 0$ for all sufficient small $\varepsilon > 0$ and all unit vectors $\vec{u}$, then f has a local max @ $\vec{p}$

$\rightarrow$

← NB: this is too hard to apply in this class for problems ... Is there Anything Better?

Yes, but there are failing conditions

To get a Second derivative test we need

$$D = \det \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

$$= f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx}$$

$$= f_{xx} \cdot f_{yy} - (f_{xy})^2$$

takes place of the second derivative

Second derivative test: This only works as stated for $f(x,y)$ (2 variables)

Let $F(x,y)$ be differentiable at cp $(\vec{p})$

① if $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) \cdot f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$ then $\vec{p}$ is a local min pt. of f

② If $f_{xx}(\vec{p}) < 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) \cdot f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$ then $\vec{p}$ is a local max pt. of f

③ if $D(\vec{p}) = f_{xx}(\vec{p}) \cdot f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 < 0$, then $\vec{p}$ is a saddle point of f

(locally, f looks like a hyperbolic paraboloid at $\vec{p}$)